Last Time: Space Curves

$$\vec{r}: I \to \mathbb{R}^n$$

↳ Interval

Recall: Limit of space curve is the component-wise limit

Ex: Compute $\lim\limits_{t \to \infty} \left\langle \frac{1+t^2}{1-t^2}, \arctan(t), \frac{1-e^{-2t}}{t} \right\rangle$

Soln: $\lim\limits_{t \to \infty} x(t) = \lim\limits_{t \to \infty} \frac{1+t^2}{1-t^2} = \lim\limits_{t \to \infty} \frac{\frac{1}{t^2}+1}{\frac{1}{t^2}-1} = \frac{0+1}{0-1} = -1$

$\lim\limits_{t \to \infty}$ and $y(t) = \lim\limits_{t \to \infty} \arctan(t) = \frac{\pi}{2}$

$\lim\limits_{t \to \infty} z(t) = \lim\limits_{t \to \infty} \frac{1-e^{-2t}}{t} = 0$

Hence, $\lim\limits_{t \to \infty} \left\langle \frac{1+t^2}{1-t^2}, \arctan(t), \frac{1-e^{-2t}}{t} \right\rangle = \left\langle -1, \frac{\pi}{2}, 0 \right\rangle$

Defn: A space curve $\vec{r}(t)$ is continuous at time $t=a$ if
$$\lim_{t \to \infty} \vec{r}(t) = \vec{r}(a)$$

| "cts:" |
| "Continuous" |

NB: A curve is continuous at time $t=a$ if and only if each of its components is continuous at time $t=a$.

Ex: When is $\vec{r}(t) = \left\langle \frac{1+t^2}{1-t^2}, \arctan(t), \frac{1-e^{-2t}}{t} \right\rangle$ cts?

Soln: $x(t) = \frac{1+t^2}{1-t^2}$ is cts at t iff $t \neq \pm 1$, i.e, $t \in (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$

$y(t) = \arctan(t)$ is cts for $t \in (-\infty, \infty)$

$z(t) = \frac{1-e^{-2t}}{t}$ is cts on $t \in (-\infty, 0) \cup (0, \infty)$

∴ $\vec{r}(t)$ is cts for $t \in (-\infty, -1) \cup (-1, 0) \cup (0, 1) \cup (1, \infty)$
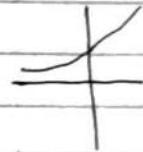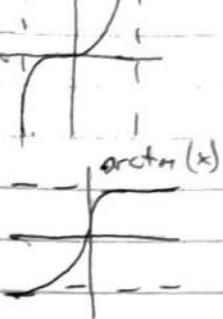
Derivatives

Defn. The derivative of space curve $\vec{r}(t)$ at the time $t=a$ is
$$\vec{r}'(a) = \frac{d\vec{r}}{dt}\Big|_{t=a} = \lim_{h \to 0} \frac{\vec{r}(a+h) - \vec{r}(a)}{h}$$

Ex: Compute $\vec{r}'(t)$ for $\vec{r}(t) = \langle t, t^3, \sqrt{t} \rangle$

Soln. $\vec{r}'(t) = \lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h} = \lim_{h \to 0} \frac{1}{h}\langle t+h, (t+h)^3, \sqrt{t+h} \rangle$

$= \lim_{h \to 0} \frac{1}{h} \langle h, 3t^2 h + 3th^2 + h^3, \sqrt{t+h} - \sqrt{t} \rangle$

$= \lim_{h \to 0} \langle 1, 3t^2 + 3th + h^2, \frac{\sqrt{t+h} - \sqrt{t}}{h} \rangle$

$= \langle \lim_{h \to 0} 1, \lim_{h \to 0} 3t^2 + 3th + h^2, \lim_{h \to 0} \frac{\sqrt{t+h} - \sqrt{t}}{h} \rangle$

$= \langle 1, 3t^2, \frac{1}{2} t^{-\frac{1}{2}} \rangle$

because: $\lim_{h \to 0} \frac{\sqrt{t+h} - \sqrt{t}}{h} = \lim_{h \to 0} \left(\frac{\sqrt{t+h} - \sqrt{t}}{h} \cdot \frac{\sqrt{t+h} + \sqrt{t}}{\sqrt{t+h} + \sqrt{t}} \right) = \lim_{h \to 0} \frac{1}{\sqrt{t+h} + \sqrt{t}}$

$= \frac{1}{\sqrt{t} + \sqrt{t}} = \frac{1}{2\sqrt{t}}$

What's really going on ($n=2$): $\vec{r}(t) = \langle x(t), y(t) \rangle$

$\vec{r}'(t) = \lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$

$= \lim_{h \to 0} \langle \frac{x(t+h) - x(t)}{h}, \frac{y(t+h) - y(t)}{h} \rangle$

$= \langle \lim_{h \to 0} \frac{x(t+h) - x(t)}{h}, \lim_{h \to 0} \frac{y(t+h) - y(t)}{h} \rangle$

$= \langle x'(t), y'(t) \rangle$ Point: The derivative is again a comparative operation

I chose the wrong day to take notes

Prop (Properties of Space-Curve Derivative):
Let $\vec{r}(t)$ and $\vec{s}(t)$ be space curves in $\mathbb{R}^n$ and let $c(t)$ be a scalar function.

① $\frac{d}{dt}\left[\vec{r}(t) + \vec{s}(t)\right] = \vec{r}'(t) + \vec{s}'(t)$ ⟶ Sum Rule

② $\frac{d}{dt}\left[c(t)\vec{r}(t)\right] = c'(t)\vec{r}(t) + c(t)\vec{r}'(t)$ ⟶ Scalar Product Rule

③ $\frac{d}{dt}\left[\vec{r}(t) \cdot \vec{s}(t)\right] = \vec{r}'(t) \cdot \vec{s}(t) + \vec{r} \cdot \vec{s}'(t)$ ⟶ Dot Product Rule

(n=3) ④ $\frac{d}{dt}\left[\vec{r}(t) \times \vec{s}(t)\right] = \vec{r}'(t) \times \vec{s}(t) + \vec{r}(t) \times \vec{s}'(t)$ Cross Product Rule

⑤ $\frac{d}{dt}\left[\vec{r}(c(t))\right] = \vec{r}'(c(t)) \, c'(t)$ ⟶ Chain Rule

Exercise: Verify each of the properties for space curves in $\mathbb{R}^3$.

Terminology: The _tangent vector_ to space curve $\vec{r}(t)$ a time $t=a$ is $\vec{r}'(a)$.

The _unit tangent vector_ at $t=a$ is $\frac{\vec{r}'(t)}{|\vec{r}'(a)|}$.

The _speed_ of $\vec{r}(t)$ at $t=a$ is $|\vec{r}'(a)|$.

Exercise: Prove that if $\vec{r}(t)$ has constant speed, then $\vec{r}(t)$ is orthogonal to $\vec{r}(t)$ for all t.

Integrals:
Defn. The definite integral of space curve from $t=a$ to b is
$$\int_{t=a}^{b} \vec{r}(t)\,dt = \int_{t=a}^{b} \langle x(t), y(t), z(t)\rangle\,dt = \left\langle \int_{t=a}^{b} x(t)\,dt, \int_{t=a}^{b} y(t)\,dt, \int_{t=a}^{b} z(t)\,dt\right\rangle$$

Interpretation: the displacement vector (of calc.1)

Arc Length

The arc length of space curve $\vec{r}(t)$ from $t = a$ to b is

$$s = \int_{t=a}^{b} |\vec{r}'(t)|\, dt$$
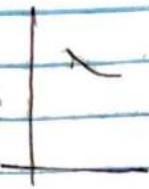
bad vs. better
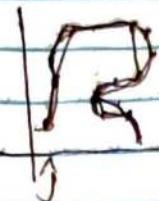
limit of secant lines.. =
tangent lines

"Piecewise-linear approx. of curve"
↳ approx. length